Designing coupling that guarantees synchronization between identical chaotic systems

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We examine synchronization between identical chaotic systems. A rigorous criteria is presented which, if satisfied, guarantees that the coupling produces linearly stable synchronous motion. The criteria can also be used to design couplings that lead to stable synchronous motion. Analytical results from a dynamical system are presented.

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Synchronization between chaotic systems has been the subject of many theoretical papers over the last few years. It has also been experimentally observed in many systems [1–9]. Despite these efforts many key issues remain open, and there are few rigorous results that ensure the stability of synchronous chaotic motion. In most cases rigorous results are obtained using Lyapunov functions [10–13]. Unfortunately, this method is not a regular approach and, in practice, it can only be applied to particular examples. The other rigorous approach is that of Ashwin et. al. [14]. To apply this approach one must show that all normal Lyapunov exponents are negative for all measures of the dynamics. For typical dynamical systems this leads to an intensive numerical analysis.

There are a few special types of coupling between nonlinear systems where rigirous analysis of the stability of synchronization is straightforward. One type is when the coupling transforms the driven system into a stable linear system with time dependent driving. A second is when the coupling is diagonal between all of the variables [1]. In many practical cases these types of coupling can't be achieved. Thus, the state of the art does not give a practical answer to the following important question: Given an arbitrary dynamical system how can one *design* a physically available coupling scheme that is guaranteed to produce stable synchronous chaotic motion?

This paper examines synchronization between identical systems with drive/response coupling. The major result is a rigorous criteria which, if satisfied, guarantees linearly stable synchronous motion. More importantly, the criteria can be used to *design* couplings that produce stable synchronized behavior. The criteria only uses knowledge of the uncoupled dynamics, and many of the important calculations can be performed analytically. Furthermore, the linearized stability equations we examine arise in many other problems that have recently appeared in the literature. A discussion of this last issue is in our longer manuscript [15].

Drive response synchronization between identical systems is modeled by

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x};t) \tag{1}$$

$$\frac{d\mathbf{y}}{dt} = \mathbf{F}(\mathbf{y};t) + \mathbf{E}(\mathbf{x}-\mathbf{y}), \tag{2}$$

where $\mathbf{x}$ is driving dynamics, $\mathbf{y}$ is the response dynamics, and $\mathbf{E}$ is a vector function representing the coupling. For these equations $\mathbf{x}$, $\mathbf{y} \in I\!R^d$ and $\mathbf{E}(\mathbf{0}) = \mathbf{0}$. Synchronization occurs on an invariant manifold given by $\mathbf{x} = \mathbf{y}$. Obviously, if the coupling strength is below some critical threshold then stable synchronous motion will not occur. For some $(\mathbf{F}, \mathbf{E})$ pairs stable synchronous motion occurs only within a finite range of coupling strengths while for others synchronization is never stable.

If one defines deviations from synchronization by $\mathbf{w} \equiv \mathbf{y} - \mathbf{x}$ then Eqs. (1) and (2) lead to the following linearized equation for motion transverse to the synchronization manifold

$$\frac{d\mathbf{w}}{dt} = [\mathbf{DF}(\mathbf{x}) - \mathbf{DE}(\mathbf{0})]\,\mathbf{w}. \tag{3}$$

In this equation $\mathbf{DF}(\mathbf{x})$ is the Jacobian of $\mathbf{F}$ evaluated on the driving trajectory, $\mathbf{x}$, and $\mathbf{DE}(\mathbf{0})$ is the Jacobian of $\mathbf{E}$ evaluated at $\mathbf{0}$. The synchronization manifold is linearly stable if $\lim_{t\to\infty} \|\mathbf{w}(t)\| = 0$ for all possible driving trajectories $\mathbf{x}(t)$ associated with the chaotic attractor of the driving system.

To determine the behavior of $\mathbf{w}(t)$ in this limit divide $\mathbf{DF}(\mathbf{x}) - \mathbf{DE}(\mathbf{0})$ into a time independent part, $\mathbf{A}$, and an explicitly time dependent part, $\mathbf{B}$,

$$\mathbf{DF}(\mathbf{x}) - \mathbf{DE}(\mathbf{0}) \equiv \mathbf{A} + \mathbf{B}(\mathbf{x}).$$

(The nonuniqueness of this decomposition will be resolved later.) Assume $\mathbf{A}$ can be diagonalized, transform to the coordinate system defined by the eigenvectors of $\mathbf{A}$, and rewrite the linearized equations of motion as the following integral equation [15]

$$\mathbf{z}(t) = \mathbf{U}(t,t_0)\mathbf{z}(t_0) + \int_{t_0}^{t} \mathbf{U}(t,s)\left[\mathbf{P}^{-1}\mathbf{B}[\mathbf{x}(s)]\mathbf{P}\right]\mathbf{z}(s)\,ds. \tag{4}$$

In this equation $\mathbf{z} \equiv \mathbf{P}^{-1}\mathbf{w}$ where $\mathbf{P} \equiv [\hat{\mathbf{e}}_1\,\hat{\mathbf{e}}_2 \cdots \hat{\mathbf{e}}_d]$ and $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\cdots \hat{\mathbf{e}}_d$ are the eigenvectors of $\mathbf{A}$. The ordering of

the eigenvectors is given by the corresponding eigenvalues, $\Re[\Lambda_1] \geq \Re[\Lambda_2] \geq \cdots \geq \Re[\Lambda_d]$, where $\Re[\Lambda]$ is the real part of Λ. Also, $\mathbf{U}(t,t_0) \equiv \exp[\mathbf{D}(t-t_0)]$ is a time evolution operator, where $\mathbf{D} \equiv \mathbf{P}^{-1}\mathbf{AP}$ is diagonal by assumption.

Linear stability of the synchronization manifold is determined by $\|\mathbf{z}(t)\|$ in the $t \to \infty$ limit. If one uses norms to convert Eq. (4) into an inequality, and applies Gronwall's theorem, then one can define the following decomposition [15]

$$\mathbf{A} = \langle \mathbf{DF} \rangle - \mathbf{DE} \tag{5}$$

$$\mathbf{B} = \mathbf{DF} - \langle \mathbf{DF} \rangle \,, \tag{6}$$

where $\langle \bullet \rangle$ denotes a time average along the driving trajectory. In terms of this decomposition the criteria for linear stability of synchronous motion is [15]

$$-\Re[\Lambda_1] > \left\langle \|\mathbf{P}^{-1} \left[\mathbf{DF}(\mathbf{x}) - \langle \mathbf{DF} \rangle\right] \mathbf{P}\| \right\rangle . \tag{7}$$

Equations (5)–(7) are our major results. They represent definitions and conditions that indicate when synchronous motion along a particular driving trajectory is *guaranteed* to be stable to small perturbations in directions transverse to the synchronization manifold. The criterion is rigorous and sufficient. However, because it is based on norms it is not necessary. Indeed, numerical experiments indicate that it tends to overestimate the necessary coupling strengths [15]. Also, since the integral in Eq. (7) is positive semi-definite the inequality can't be satisfied unless $\Re[\Lambda_1] < 0$. This condition is reminiscent of the discussion of conditional Lyapunov exponents found in previous references.

The decomposition in Eqs. (5) and (6) is optimal in the sense that it minimizes the right hand side of Eq. (7). We speculate that minimizing this integral gives one the chance at satisfying the inequality. Furthermore, by inserting Eq. (6) into a Volterra expansion of Eq. (4) one can show that, to second order, the criteria for linear stability is $\Re[\Lambda_1] < 0$ [15]. For any other decomposition this approximate stability criteria will be correct to only first order. Finally, for this decomposition Eq. (7) reduces to $\Re[\Lambda_1] < 0$ for fixed points, a result that will not hold for other decompositions.

Equations (5)–(7) depend explicitly on the measure of the driving trajectory. Gupte and Amaritkar examined synchronization using unstable periodic orbits as driving trajectories [16]. This, and later papers, show that, for fixed coupling strength, different driving trajectories have different stabilities [14,15,17]. Recently, Hunt and Ott [18] numerically examined time averages on different measures of a chaotic dynamical system and found that they tend to assume their largest values on unstable periodic orbits with the shortest periods. This behavior is also discussed in Ref. [14].

Given these observations we conjecture that, in many practical cases, the unstable fixed points of the driving system will be the first measures on the synchronization manifold to go linearly unstable as the coupling strength is changed. Thus, these trajectories should be the first measure to check for linear instability. This conjecture is examined in our longer paper and is found to be true for the examples studied [15].

Equation (7) has a geometrical interpretation which can be used to *design* couplings that yield stable synchronous motion. The elements of $\mathbf{DE}(\mathbf{0})$ define a parameter space and each side of Eq. (7) defines a function in this parameter space. Thus, $\Sigma_{\mathbf{x}}$ and Σ_Λ, respectively defined by $\left\langle \|\mathbf{P}^{-1} \left[\mathbf{DF}(\mathbf{x}) - \langle \mathbf{DF} \rangle\right] \mathbf{P}\| \right\rangle = \text{const.} \equiv C_1$ and $-\Re[\Lambda_1] = \text{const.} \equiv C_2$, are families of surfaces in this parameter space. The boundary of the portion of the parameter space that yields linearly stable synchronization is the intersection of these families of surfaces. By choosing the elements of $\mathbf{DE}(\mathbf{0})$ on portions of Σ_Λ that are "above" $\Sigma_{\mathbf{x}}$ one insures that the poles of $\mathbf{A}$ are sufficiently far into the left half plane to insure stability. Thus, designing a coupling is similar to pole placement in control theory [19].

As an example we present an analysis of the following dynamical system studied by Ott and Sommerer [17]

$$\begin{aligned}
\frac{dx}{dt} &= v_x \\
\frac{dv_x}{dt} &= -\nu v_x + 4x\left(1-x^2\right) + y^2 + f_0 \sin(\omega t) \\
\frac{dy}{dt} &= 2v_y \\
\frac{dv_y}{dt} &= -\nu v_y - 2y\left(x-p\right) - 4ky^3
\end{aligned} \tag{8}$$

where $\nu = 0.05$, $f_0 = 2.3$, $\omega = 3.5$, $k = 0.0075$ and $p = -1.5$. Originally, Ott and Sommerer examined the stability of the invariant manifold defined by $y = v_y = 0$. Their results indicate that for these parameter values motion on this manifold is chaotic, the manifold itself is unstable, and only one stable attracting set exists in $\mathbb{R}^4$.

As before, $\mathbf{x}$ denotes the driving system and $\mathbf{y}$ denotes the response system. In principle the driving trajectory, $\mathbf{x} = [x, v_x, y, v_y] \in \mathbb{R}^4$. However, for this example we consider a driving trajectory restricted to the manifold examined by Ott and Sommerer. For this type of driving $\mathbf{DF}(\mathbf{x})$ assumes a block diagonal form. If we use block diagonal coupling then Eq. (3) decomposes into motion parallel to, and perpendicular to the manifold examined by Ott and Sommerer.

For perpendicular motion

$$\frac{d\mathbf{w}^{(\perp)}}{dt} = \left[\mathbf{DF}^{(\perp)}(\mathbf{x}) - \mathbf{DE}^{(\perp)}(\mathbf{0})\right] \mathbf{w}^{(\perp)}, \tag{9}$$

where

$$\mathbf{DF}^{(\perp)}(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ g^{(\perp)}(\mathbf{x}) & -\nu \end{bmatrix}$$

$$\mathbf{DE}^{(\perp)}(\mathbf{0}) = \begin{bmatrix} \epsilon_1^{(\perp)} & \epsilon_4^{(\perp)} \\ \epsilon_3^{(\perp)} & \epsilon_2^{(\perp)} \end{bmatrix},$$

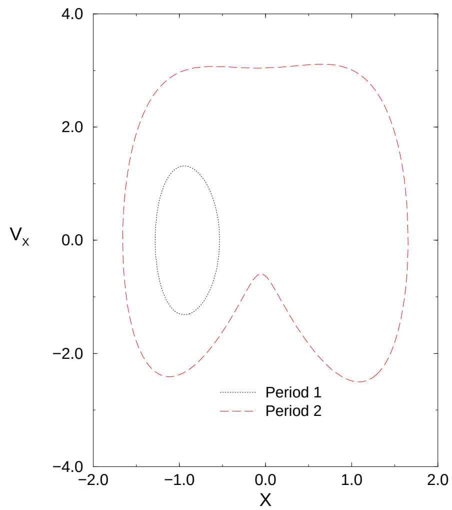

FIG. 1. Period 1 and period 2 orbits of the Ott-Sommerer model.

and $g^{(\perp)}(\mathbf{x}) \equiv -2(x-p)$. (Equation (9) is the same linear stability equation studied by Ott and Sommerer, Eqs. (7) and (8) in Ref. [17]). An equation similar to Eq. (9) involving $\mathbf{w}^{(\parallel)}$, $\mathbf{DF}^{(\parallel)}$, $\mathbf{DE}^{(\parallel)}$, and $g^{(\parallel)}(\mathbf{x}) \equiv 4(1-3x^2)$ exists for motion parallel to the manifold. (For the remainder of this letter we drop the $\perp$ and $\parallel$ superscripts, and trust the reader to perform calculations in *both* the perpendicular and parallel subspaces.)

It is easy to show that the eigenvalues of $\mathbf{A}$ are

$$\Lambda_\pm = \frac{-(\nu+\epsilon_1+\epsilon_2)}{2} \pm \frac{1}{2}\left[(\nu+\epsilon_2-\epsilon_1)^2 + 4(1-\epsilon_4)(\langle g\rangle - \epsilon_3)\right]^{1/2}. \quad (10)$$

If $\Lambda_\pm$ are complex then $-\Re[\Lambda_1]$ can be made arbitrarily large by increasing ϵ_1 and/or ϵ_2. The case for real $\Lambda_\pm$ is more complicated, however, numerical results indicate that $-\Re[\Lambda_1]$ is maximized when $\Lambda_\pm$ are complex [15]. Because, $\left\langle \left\| \mathbf{P}^{-1}\left[\mathbf{DF}(\mathbf{x}) - \langle \mathbf{DF}\rangle\right]\mathbf{P}\right\| \right\rangle$ diverges as $\Lambda_\pm$ transitions from real to complex ϵ's associated with this transition should to be avoided. These observations suggest that in order to satisfy the condition for linear stability of the synchronization manifold one should choose ϵ's so that $\Lambda_\pm$ are complex with imaginary parts that are not near zero.

It is possible to show that if $\Lambda_\pm$ are complex then the condition for linear stability of synchronous motion is

$$\nu + \epsilon_1 + \epsilon_2 > 4C\,\langle|\Delta g|\rangle\,, \quad (11)$$

where

$$C \equiv \left[\frac{-(1-\epsilon_4)^2}{(\nu+\epsilon_2-\epsilon_1)^2 + 4(1-\epsilon_4)(\langle g\rangle - \epsilon_3)}\right]^{1/2}, \quad (12)$$

Measure Type	$\langle g^{(\perp)}\rangle$	$\langle g^{(\parallel)}\rangle$	$\langle\|\Delta g^{(\perp)}\|\rangle$	$\langle\|\Delta g^{(\parallel)}\|\rangle$
Period 1	-1.223	-6.307	0.4769	5.142
Period 2	-3	-7.767	1.678	10.30
SBR	-3	-7.038	1.714	7.856

TABLE I.

and $\Delta g(\mathbf{x}) \equiv g(\mathbf{x}) - \langle g\rangle$. Equations (10)–(12), and the conjecture that the ϵ's should be chosen so that $\Lambda_\pm$ are complex, is an *analytic solution* to the rigorous criteria for synchronization. Equating C to a real positive constant, in effect, selects surfaces from the families $\Sigma_{\mathbf{x}}$ and Σ_Λ. Each driving trajectory, $\mathbf{x}$, corresponds to a different surface.

Since Eqs. (8) do not have fixed points we examined the SBR measure, and the measures associated with the periodic orbits shown in Fig. 1 (the SBR measure is shown in Fig. 1 of Ref [17]). Table I shows numerically calculated values for $\langle g\rangle$ and $\langle|\Delta g|\rangle$.

We now explicitly examine several types of driving. The first is diagonal driving. It is the first of the special cases where rigirous results are straightforward [1,5]. Diagonal driving uses all components of $\mathbf{x}$ and chooses $\epsilon_3 = \epsilon_4 = 0$, $\epsilon_1 = \epsilon_2 \equiv \epsilon$ [1,5]. The parameter space is the real line, $I\!R$. For this type of driving $\Lambda_\pm$ are complex on each of our measures for all values of ϵ. Since C is independent of ϵ its value is fixed, $\Sigma_{\mathbf{x}}$ is the entire parameter space, and Σ_Λ is a family of points in $I\!R$. Thus, for a particular driving trajectory the boundary for linear stability of the synchronization manifold (the intersection of Σ_Λ with $\Sigma_{\mathbf{x}}$) is given by a point in $I\!R$. The rigorous criteria for synchronization, Eq. (11), is

$$\epsilon > -\frac{\nu}{2} + 2\,\langle|\Delta g|\rangle \left[\frac{-1}{\nu^2 + 4\,\langle g\rangle}\right]^{1/2}.$$

Driving via postion uses only the position variables, x and y. The simplest example is $\epsilon_2 = \epsilon_3 = \epsilon_4 = 0$ and the parameter space is again $I\!R$. It is useful to define new parameters $u \equiv \epsilon_1 + \nu$ and $w \equiv 1/C$. In terms of u and w Eqs. (11) and (12) are

$$uw > 4\,\langle|\Delta g|\rangle$$
$$-4\,\langle g\rangle = (u-2\nu)^2 + w^2.$$

If $\langle g\rangle < 0$ then these equations define a hyperbola and a circle, respectively. It is straightforward to show that the circle does not intersect the hyperbola on the measures we have examined. Thus, for these driving trajectories the rigorous condition for synchronization can not be satisfied. (This does not mean that stable synchronization *will not* result from this type of driving. It only means that our analysis can not *guarantee* that stable synchronization will result from this type of driving [15].)

Another example of this type of driving uses the positions to drive both the position *and* the velocity equations. (This can sometimes synchronize systems when

simple driving via position does not produce synchronization [20].) For this type of driving, $\epsilon_2 = \epsilon_4 = 0$ and the parameter space is $I\!R^2$. Also, $\Lambda_\pm$ are complex for all $C > 0$, while Eqs. (11) and (12) become

$$\epsilon_1 > -\nu + 4C \langle |\Delta g| \rangle$$
$$\epsilon_3 = \frac{1}{4}(\nu - \epsilon_1)^2 + \left[\langle g \rangle + \frac{1}{4C^2} \right].$$

These equations define a line and a parabola in $I\!R^2$. For any driving trajectory the line and the parabola are guaranteed to intersect. Therefore, synchronization to *any* trajectory, $\mathbf{x}$, is guaranteed to be linearly stable for coupling strengths on the parabola whose ϵ_1 value is larger than the one associated with the intersection.

Driving via velocity uses only the velocity variables, v_x and v_y. As an example, let the velocities drive both the position and the velocity equations. Thus, $\epsilon_1 = \epsilon_3 = 0$, the parameter space is $I\!R^2$, and $\Lambda_\pm$ are complex for $C > 0$. If we define new parameters $u \equiv \nu + \epsilon_2$ and $w \equiv (\epsilon_4 - 1)/C$ then Eqs. (11) and (12) become

$$u > 4C \langle |\Delta g| \rangle$$
$$(2C \langle g \rangle)^2 = u^2 + (2C \langle g \rangle - w)^2 .$$

These equations define a line and a circle, respectively. It is straightforward to show that the circle does not intersect the line on the measures we have examined. Therefore, the rigorous condition for synchronization can not be satisfied on these orbits.

In this paper we investigated the linear stability of the invariant manifold associated with synchronous behavior between coupled chaotic systems. Although we explicitly examined unidirectional coupling our results are valid for bidirectional coupling, and for determining the linear stability of invariant manifolds within a chaotic system [15].

Our major result is the rigorous criteria of Eqs. (5)–(7). When they are satisfied linear stability of synchronous motion is guaranteed. The criteria depends on the measure of the driving dynamics and can yield different results for different driving trajectories. The criteria can also be used to design couplings that produce synchronization between coupled systems.

In closing this letter we discuss how noise and nonlinear effects influence our results. Assume the driving trajectory is a fixed point, $\mathbf{x}_*$, and that Eq. (7) is satisfied for $\epsilon > \epsilon_*$ (Equivalently, a period 1 orbit evaluated on a surface of section.) For this case the fixed point undergoes a co-dimension one bifurcation (either pitchfork or transcritical) at $\epsilon = \epsilon_*$. Linear stability analysis does not take into account the unstable trajectories near $\mathbf{x}_*$ when $\epsilon \gtrsim \epsilon_*$. For arbitrarily small noise amplitude there exists a range of ϵ values near ϵ_* where the noise will eventually push the response system beyond one of the unstable orbits. When this occurs the response system is forced to seek out an attracting state away from the synchronization manifold. Also, nonlinear effects could cause an unstable orbit to approach $\mathbf{x}_*$ for some ϵ far from ϵ_*. If this occurs then small noise levels can also result in a loss of synchronization.

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[1] Fujisaka, H. and T. Yamada, *Prog. Theor. Phys.* **69**, 32 (1983).
[2] Pecora, L. M. and T. L. Carroll, *Phys. Rev. Lett.* **64**, 821 (1990); L. M. Pecora and T. L. Carroll, *Phys. Rev.* **44A** , 2374 (1991).
[3] Brown, R., N. F. Rulkov and N. B. Tufillaro, *Phys. Rev.* **50E**, 4488 (1994).
[4] Pecora, L. M., *Chaos in Communications, SPIE Proceedings, San Diego, CA, 1993*, 2038, 2-25 (SPIE-The International Society for Optical Engineering,Bellingham, WA)
[5] Heagy, J. F., T. L. Carroll and L. M. Pecora, *Phys. Rev.* **50E**, 1874 (1994).
[6] Feldmann, U., *Synchronization of Chaotic Systems* Ph.D. dissertation, Faculty of Electrical Engineering, Technical University of Dresden, July 1995.
[7] Sushchik, M. M., *Synchronized chaotic oscillations*, Ph.D. dissertation, Physics Faculty, University of California, San Diego, July 1996.
[8] de Sousa Vieira, M., A. L. Lichtenberg and M. A. Lieberman, *Phys. Rev.* **46A**, R7359 (1992).
[9] Gauthier, D. J. and J. C. Bienfang, *Phys. Rev. Lett.* **77**, 1751 (1996).
[10] He. R. and P. G. Vaidya, *Phys. Rev.* **46A**, 7387 (1992).
[11] Rodrigues, H. M., *Uniform ultimate boundedness and synchronization* preprint CDSNS94-160 Georgia Institute of Technology.
[12] Rulkov, N. F. et.al., *Int. J. Bif. Chaos* **2**, 669 (1992).
[13] Wu, C. W and L. O. Chua, *Int. J. Bif. Chaos* **4**, 979 (1994).
[14] Ashwin, P., J. Buescu and I. Stewart, *Phys. Letts.* **193A**, 126 (1994); P. Ashwin, J. Buescu and I. Stewart, *Nonlinearity* **9**, 703 (1996).
[15] Brown, R. and N. F. Rulkov, "Synchronization of chaotic systems: Transverse stability of trajectories in invariant manifolds" submitted for publication.
[16] Gupte, N. and R. E. Amritkar, *Phys. Rev.* **48E**, R1620 (1993).
[17] Ott, E. and J. C. Sommerer, *Phys. Lett.* **188A**, 39 (1994).
[18] Hunt, B. R. and E. Ott, *Phys. Rev. Lett.* **76**, 2254 (1996).
[19] Sontag, E. D. *Mathematical Control Theory* (Springer-Verlag, New York, 1990).
[20] Ding, M.-Z. and E. Ott, *Phys. Rev.* **49E**, R945 (1994).